Quadrocopter Pole Acrobatics - Online Appendix

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I. Introduction

In this online appendix, we present the full derivations of the pole dynamics and the nominal throw maneuver. The section numbering is identical with the original paper.

II. System Dynamics

If not explicitly mentioned otherwise, all vectors will be expressed in a stationary inertial frame I. For the ease of notation, vectors are expressed as n-tuples $(x_1, x_2, \dots)$, with dimension and stacking clear from context.

B. Pole

The pole (see Fig. 1) is modelled as a thin rod of length $2L$, mass m_p, and an inertia tensor $\bar{\Theta}$ with respect to its center of mass expressed in the pole fixed frame K:

$$\bar{\Theta} = \begin{bmatrix} \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

with $\Theta = \frac{1}{3} m_p L^2$.

We define the vectors ${}_I e_x^K$, ${}_I e_y^K$, and n to be the coordinate axis of the pole-fixed frame K, expressed in the inertial frame I. We furthermore define ${}_K e_x^I$, ${}_K e_y^I$, and ${}_K e_z^I$ to be the coordinate axis that span the inertial frame I, expressed in the pole-fixed frame K. The rotation matrix ${}^I_K R$ that maps a vector from K to I is then given by

$$ {}^I_K R = \begin{bmatrix} {}_I e_x^K & {}_I e_y^K & n \end{bmatrix} = \begin{bmatrix} {}_K e_x^I \\ {}_K e_y^I \\ {}_K e_z^I \end{bmatrix} \tag{2}$$

and its inverse is given by

$$ {}^K_I R = \begin{bmatrix} {}_K e_x^I & {}_K e_y^I & {}_K e_z^I \end{bmatrix} = \begin{bmatrix} {}_I e_x^K \\ {}_I e_y^K \\ n \end{bmatrix}. \tag{3}$$

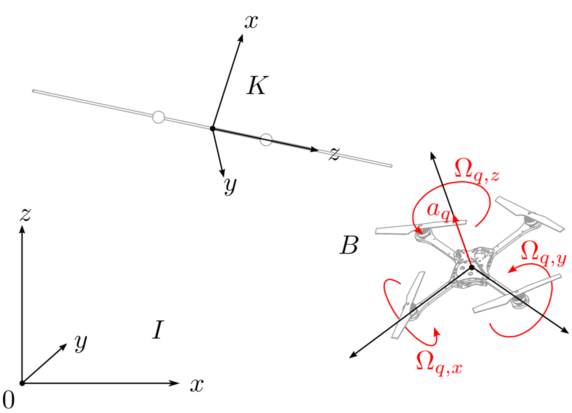

Fig. 1. Inertial reference frame I and the quadrocopter and pole with their body-fixed coordinate frames B and K, respectively. The quadrocopter control inputs are a mass-normalized thrust a_q along the body-fixed z-axis and the angular rates $(\Omega_{q,x}, \Omega_{q,y}, \Omega_{q,z})$.

The inertia tensor expressed in the inertial frame I then simplifies to:

$$
\begin{aligned}
{}^I_K R \bar{\Theta} {}^K_I R &= \Theta {}^I_K R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} {}^K_I R && (4)\\
&= \Theta {}^I_K R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} {}_I e_x^K \\ {}_I e_y^K \\ n \end{bmatrix} && (5)\\
&= \Theta \begin{bmatrix} {}_K e_x^I \\ {}_K e_y^I \\ {}_K e_z^I \end{bmatrix} \begin{bmatrix} {}_I e_x^K \\ {}_I e_y^K \\ 0 \end{bmatrix} && (6)\\
&= \Theta \begin{bmatrix} {}_K e_x^I \\ {}_K e_y^I \\ {}_K e_z^I \end{bmatrix} \left(\begin{bmatrix} {}_K e_x^I & {}_K e_y^I & {}_K e_z^I \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} \right) && (7)\\
&= \Theta \left(\mathbf{1} - \begin{bmatrix} {}_I e_x^K & {}_I e_y^K & n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} \right) && (8)\\
&= \Theta \left(\mathbf{1} - n n^{\mathrm{T}} \right). && (9)
\end{aligned}
$$

This expression is used in (18) in the paper.

1) Free flight: The pole mass center position is represented by $p_p = (x_p, y_p, z_p)$ and its reduced attitude [1] is expressed using the unit vector n pointing along the pole z-axis. During flight, aerodynamic drag and gravity act upon the pole. Due to the small cross section of the pole along its z-axis and the comparatively large cross section orthogonal to it, the pole's drag properties depend heavily on its orientation. Consequently, the drag force is split into two components: a force in the direction of the pole z-axis and a force in its xy-plane. The drag forces themselves are modelled proportionally to the speed squared:

$$F_{drag,z} = -c_z \int_{-L}^{L} \|\dot{p}_z(\xi)\| \dot{p}_z(\xi) d\xi \tag{10}$$

$$F_{drag,xy} = -c_{xy} \int_{-L}^{L} \|\dot{p}_{xy}(\xi)\| \dot{p}_{xy}(\xi) d\xi, \tag{11}$$

where $c_{(\cdot)}$ is the drag coefficient and $\dot{p}(\xi)$ is the velocity of a point on the pole at a distance of ξ from the center. Given the pole center velocity $\dot{p}_p$ and its angular rate Ω_p, $\dot{p}(\xi)$ is given by:

$$\dot{p}(\xi) = \dot{p}_p + \Omega_p \times (\xi n). \tag{12}$$

The pole-fixed z- and xy-components of the velocity can then be calculated:

$$\dot{p}_z(\xi) = \left(\dot{p}(\xi)^{\mathrm{T}} n \right) n \tag{13}$$

$$\dot{p}_{xy}(\xi) = \dot{p}(\xi) - \dot{p}_z(\xi). \tag{14}$$

The torque caused by the aerodynamic drag follows analogous to (11):

$$M_{drag} = -c_{rot} \int_{-L}^{L} (\xi n) \times \left(\|\dot{p}(\xi)\| \dot{p}(\xi) \right) d\xi. \tag{15}$$

The Newton-Euler equation for the pole in free flight yields

$$m_p \ddot{p}_p = F_{drag,z} + F_{drag,xy} - m_p g \tag{16}$$

$$\Theta \left(\mathbf{1} - n n^{\mathrm{T}} \right) \dot{\Omega}_p = M_{drag}. \tag{17}$$

It follows from (15) that no torque is induced about the pole z-axis and hence it is assumed that the angular rate about this axis remains constant, i.e. the angular acceleration along the pole-fixed z-axis is zero:

$$n^{\mathrm{T}} \dot{\Omega}_p = 0. \tag{18}$$

In this case, the rotational dynamics (17) simplify to

$$\Theta\left(\mathbf{1}-nn^{\mathrm{T}}\right)\dot{\Omega}_p = M_{drag} \tag{19}$$
$$\Theta\dot{\Omega}_p - \Theta nn^{\mathrm{T}}\dot{\Omega}_p = M_{drag} \tag{20}$$
$$\Theta\dot{\Omega}_p = M_{drag} \tag{21}$$
$$\dot{\Omega}_p = \frac{1}{\Theta}M_{drag}, \tag{22}$$

which is (19) in the paper. The dynamics for the pole in free flight can be written as

$$\dot{s}_p = f_p(s_p) \tag{23}$$

with the pole state s_p being defined as

$$s_p = (p_p, \dot{p}_p, n, \Omega_p) \tag{24}$$

and f_p containing (16), (22) and the kinematic relation

$$\dot{n} = \Omega_p \times n. \tag{25}$$

2) On quadrocopter: In the case where the pole is in contact with a quadrocopter, it is assumed that the pole is rigidly attached to the quadrocopter's mass center. The pole position then depends on the quadrocopter:

$$p_p = p_q + Ln \tag{26}$$

where we assume, without loss of generality, that n points from the quadrocopter towards the pole mass center. Subsequently, the pole attitude is parametrized by the deflection of the pole center relative to its supporting point (see Fig. 2):

$$Ln = \left(a, b, \sqrt{L^2 - a^2 - b^2}\right) \tag{27}$$

where a denotes the deflection in the inertial x-direction and b in the y-direction, respectively. The Lagrangian of the pole is given by

$$\mathscr{L} = \frac{1}{2}m_p\dot{p}_p^{\mathrm{T}}\dot{p}_p + \frac{1}{2}\Omega_p^{\mathrm{T}}\Theta\left(\mathbf{1}-nn^{\mathrm{T}}\right)\Omega_p - m_p g^{\mathrm{T}} p_p \tag{28}$$

with

$$\dot{p}_p = \left(\dot{x}_q + \dot{a}, \dot{y}_q + \dot{b}, \dot{z}_q - \frac{a\dot{a} + b\dot{b}}{\sqrt{L^2 - a^2 - b^2}}\right). \tag{29}$$

The center term $\frac{1}{2}\Omega_p^{\mathrm{T}}\Theta\left(\mathbf{1}-nn^{\mathrm{T}}\right)\Omega_p$ of (28) represents the kinetic energy when the pole rotates. Let the angular rate of the pole Ω_p be expressed as the sum of an angular rate about the pole z-axis and an angular rate in the xy-plane:

$$\Omega_p = \Omega_{p,z} + \Omega_{p,xy}. \tag{30}$$

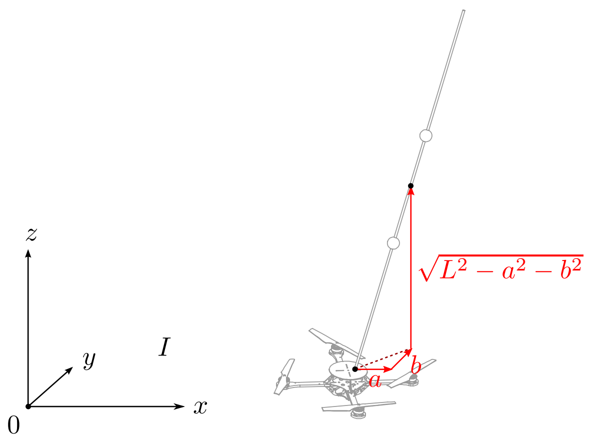

Fig. 2. Quadrocopter-pole system with the pole balanced on the quadrocopter. It is assumed that the pole is attached to the quadrocopter's center of gravity.

By definition, the following properties hold:

$$\Omega_{p,z}^{\mathrm{T}}\Omega_{p,xy} = 0 \tag{31}$$
$$n^{\mathrm{T}}\Omega_{p,xy} = 0 \tag{32}$$
$$\left(\Omega_p^{\mathrm{T}} n\right) n = \Omega_{p,z}. \tag{33}$$

Inserting (30) into the kinetic energy for the rotating pole yields

$$\begin{aligned}
\frac{1}{2}\Omega_p^{\mathrm{T}}\Theta\left(\mathbf{1} - nn^{\mathrm{T}}\right)\Omega_p &= \frac{1}{2}\Theta\left(\Omega_p^{\mathrm{T}}\Omega_p - \Omega_p^{\mathrm{T}} nn^{\mathrm{T}}\Omega_p\right) && (34)\\
&= \frac{1}{2}\Theta\left(\left(\Omega_{p,z} + \Omega_{p,xy}\right)^{\mathrm{T}}\left(\Omega_{p,z} + \Omega_{p,xy}\right) - \left(\Omega_{p,z} + \Omega_{p,xy}\right)^{\mathrm{T}} nn^{\mathrm{T}}\left(\Omega_{p,z} + \Omega_{p,xy}\right)\right) && (35)\\
&= \frac{1}{2}\Theta\left(\Omega_{p,z}^{\mathrm{T}}\Omega_{p,z} + \Omega_{p,xy}^{\mathrm{T}}\Omega_{p,xy} - \Omega_{p,z}^{\mathrm{T}}\Omega_{p,z}\right) && (36)\\
&= \frac{1}{2}\Theta\Omega_{p,xy}^{\mathrm{T}}\Omega_{p,xy}. && (37)
\end{aligned}$$

Because the pole has no moment of inertia about its z-axis, rotating about it does add kinetic energy to the system and can thus be left out. The component of the angular rate in the pole-fixed xy-plane can be recovered from (25) using the triple cross product expansion:

$$\begin{aligned}
\dot{n} &= \Omega_p \times n && (38)\\
\dot{n} \times n &= \left(\Omega_p \times n\right) \times n && (39)\\
&= \left(\Omega_p^{\mathrm{T}} n\right) n - \left(n^{\mathrm{T}} n\right)\Omega_p && (40)\\
&= \left(\left(\Omega_{p,z} + \Omega_{p,xy}\right)^{\mathrm{T}} n\right) n - \left(\Omega_{p,z} + \Omega_{p,xy}\right) && (41)\\
&= \Omega_{p,z} - \Omega_{p,z} - \Omega_{p,xy}, && (42)
\end{aligned}$$

and finally

$$\Omega_{p,xy} = n \times \dot{n}, \tag{43}$$

which is (27) in the paper. The Lagrangian of the pole (28) can thus be written as

$$\mathscr{L} = \frac{1}{2}m_p\dot{p}_p^{\mathrm{T}}\dot{p}_p + \frac{1}{2}\Theta\Omega_{p,xy}^{\mathrm{T}}\Omega_{p,xy} - m_p g^{\mathrm{T}} p_p, \tag{44}$$

which is (25) in the paper. The nonlinear dynamics are derived applying Lagrangian dynamics:

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{a}}\right) - \frac{\partial\mathscr{L}}{\partial a} = 0 \tag{45}$$
$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{b}}\right) - \frac{\partial\mathscr{L}}{\partial b} = 0, \tag{46}$$

resulting in the nonlinear equations

$$\ddot{a} = \frac{a\left(4\dot{b}^2\left(a^2 - L^2\right) - 8\dot{a}\dot{b}ab + 4\dot{a}^2\left(b^2 - L^2\right) + 3\zeta^3(g + \ddot{z}_q)\right) + 3\zeta^2\left(ab\ddot{y}_q + \left(a^2 - L^2\right)\ddot{x}_q\right)}{4L^2\zeta^2} \tag{47}$$

$$\ddot{b} = \frac{b\left(4\dot{b}^2\left(a^2 - L^2\right) - 8\dot{a}\dot{b}ab + 4\dot{a}^2\left(b^2 - L^2\right) + 3\zeta^3(g + \ddot{z}_q)\right) + 3\zeta^2\left(ab\ddot{x}_q + \left(b^2 - L^2\right)\ddot{y}_q\right)}{4L^2\zeta^2}, \tag{48}$$

where $\ddot{p}_q = (\ddot{x}_q, \ddot{y}_q, \ddot{z}_q)$ is the acceleration of the quadrocopter mass center and $\zeta = \sqrt{L^2 - a^2 - b^2}$ is the z-position of the pole relative to its supporting point. The dynamics of the combined quadrocopter-pole system can be described by

$$\dot{s}_{qp} = f_{qp}(s_{qp}, u_q), \tag{49}$$

where s_{qp} is the combined quadrocopter-pole state

$$s_{qp} = \left(s_q, a, b, \dot{a}, \dot{b}\right) \tag{50}$$

and f_{qp} contains the quadrocopter dynamics f_q and the pole dynamics (47,48). These dynamics are (30) in the paper.

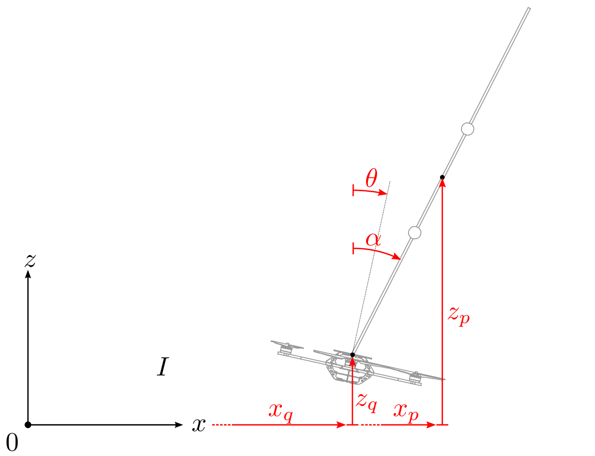

Fig. 3. A simplified two-dimensional model of the quadrocopter-pole system in the xz-plane. The pole attitude is parametrized by the tilt angle α and the quadrocopter attitude is fully described by the pitch angle θ.

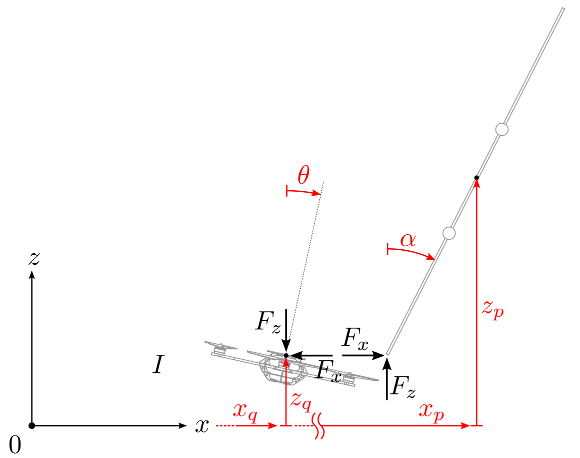

Fig. 4. Free body diagram of the two dimensional quadrocopter-pole system.

III. Nominal Maneuver Design

A. Throw Maneuver

The throwing of the pole is designed to take place in the vertical xz-plane. We therefore use a simplified two-dimensional quadrocopter-pole model with the quadrocopter Euler angles $\phi = \psi = 0$ and the pole deflection $b = 0$ (see Fig. 3). The pole attitude is parametrized by the tilt angle α

$$\alpha = \sin^{-1}\left(\frac{a}{L}\right). \tag{51}$$

The pole position (26) can then be written as

$$x_p = x_q + L\sin\alpha, \tag{52}$$
$$z_p = z_q + L\cos\alpha, \tag{53}$$

and the pole acceleration is given by

$$\ddot{x}_p = \ddot{x}_q - L\dot{\alpha}^2\sin\alpha + L\ddot{\alpha}\cos\alpha, \tag{54}$$
$$\ddot{z}_p = \ddot{z}_q - L\dot{\alpha}^2\cos\alpha - L\ddot{\alpha}\sin\alpha. \tag{55}$$

Considering the free body diagram in Fig. 4, the Newton-Euler equation for the pole yields

$$m_p\ddot{x}_p = F_x, \tag{56}$$
$$m_p\ddot{z}_p = F_z - m_p g, \tag{57}$$
$$\Theta\ddot{\alpha} = L\sin\alpha F_z - L\cos\alpha F_x, \tag{58}$$

where F_x and F_z are the contact forces from the quadrocopter that act on the pole in x- and z-direction, respectively. Because it is assumed that the pole has no impact on the quadrocopter dynamics, the quadrocopter dynamics can be obtained by evaluating (4) of the paper for the two dimensional case:

$$\ddot{x}_q = a_q\sin\theta, \tag{59}$$
$$\ddot{z}_q = a_q\cos\theta - g. \tag{60}$$

Finally, inserting the pole kinematics (54,55) and the quadrocopter dynamics (59,60) into the pole dynamics (56-58) yields the following expressions:

$$F_x = \frac{1}{4}\left(\left(1 + 3\sin^2\alpha\right)m_p a_q\sin\theta - 4m_p L\dot{\alpha}^2\sin\alpha + 3m_p a_q\cos\alpha\sin\alpha\cos\theta\right), \tag{61}$$
$$F_z = \frac{1}{4}\left(\left(1 + 3\cos^2\alpha\right)m_p a_q\cos\theta - 4m_p L\dot{\alpha}^2\cos\alpha + 3m_p a_q\cos\alpha\sin\alpha\sin\theta\right). \tag{62}$$

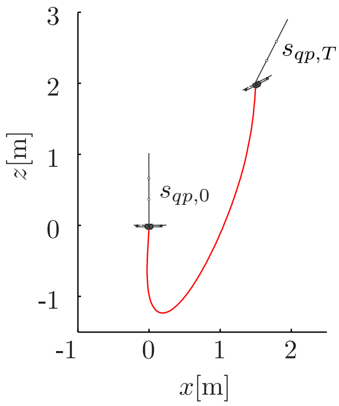

Fig. 5. Throwing trajectory for a maneuver length of $T = 2.5\,\mathrm{s}$. The final constraint $s_{qp,T}$ consists of position $(x_q, z_q) = (1.5\,\mathrm{m}, 2.0\,\mathrm{m})$, velocity $(\dot{x}_q, \dot{z}_q) = (0.3\,\mathrm{m\,s^{-1}}, 5.5\,\mathrm{m\,s^{-1}})$, attitudes $(\alpha, \theta) = (0.48\,\mathrm{rad}, -0.48\,\mathrm{rad})$, and angular rate $\dot{\alpha} = 2.5\,\mathrm{rad\,s^{-1}}$, and with the final control input $u_{q,T}$ being $(a_q, \Omega_{q,y}) = (4.48\,\mathrm{m\,s^{-2}}, 2.5\,\mathrm{rad\,s^{-1}})$.

The normal force between the quadrocopter and pole is then computed by projecting the forces F_x and F_z onto the quadrocopter z-axis:

$$F_N(s_{qp}, u_q) = \sin\theta F_x + \cos\theta F_z, \tag{63}$$

$$F_N(s_{qp}, u_q) = m_p \left(a_q - \frac{3}{4} a_q \sin^2(\alpha - \theta) - L\dot{\alpha}^2 \cos(\alpha - \theta) \right). \tag{64}$$

Equation (64) replaces (33) in the paper[1]. For the pendulum to be launched off the vehicle, $F_N < 0$ is required. We note that, as discussed in Section II of the paper, $a_q > 0$. The pole can therefore not be thrown vertically, but must be accelerated to a sufficiently high rotational rate $\dot{\alpha}$.

We generate a trajectory to maneuver the quadrocopter from a stationary starting point to a state at which the pole leaves the quadrocopter, using optimal control methods:

$$\begin{array}{ll} \text{minimize} & \int_0^T 1.5\alpha(t)^2 + \left(a_q(t) - g\right)^2 + 8\Omega_{q,y}(t)^2 dt \\ \text{subject to} & \dot{s}_{qp}(t) = f_{qp}(s_{qp}(t), u_q(t)) \\ & u_{q,min} \leq u_q(t) \leq u_{q,max} \\ & 0 \leq F_N(s_{qp}(t), u_q(t)) \\ & s_{qp}(t=0) = 0 \\ & s_{qp}(t=T) = s_{qp,T}, \end{array} \tag{65}$$

The initial constraint represents the quadrocopter hovering at the origin with the pole balanced in the equilibrium position. The final state $s_{qp,T}$ is a design parameter, that, in combination with a control input $u_{q,T}$, must satisfy $F_N(s_{qp,T}, u_{q,T}) = 0$. The cost function is designed so as to yield a smooth throwing trajectory. A solution to the optimization problem (65) for a trajectory duration $T = 2.5\,\mathrm{s}$ is shown in Fig. 5.

References

[1] N. A. Chaturvedi, A. K. Sanyal, and N. H. McClamroch, “Rigid-body attitude control,” *Control Systems, IEEE*, vol. 31, pp. 30–51, June.

[1] Note that (33) in the paper is incorrect and should be replaced with (64) of the online appendix.